Boundary behavior of holomorphic functions on the bidisk

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- $\mathbb{D}^2$ will denote the complex bidisk.
- the distinguished boundary of the complex disk is the complex torus, $\mathbb{T}^2$.
- a Schur function is a holomorphic map of the bidisk into the disk, i.e. a function $\varphi : \mathbb{D}^2 \to \mathbb{D}^-$.
- the Schur class is the family of Schur functions, denoted $\mathcal{S}_2$. 
For a point $\tau \in T^2$, a set $S \subset \mathbb{D}^2$ is said to approach $\tau$ nontangentially if there exists a $c > 0$ such that

$$\|\lambda - \tau\| \leq c(1 - \|\lambda\|).$$

A statement holds nontangentially at $\tau$ if for every set $S$ that approaches $\tau$ nontangentially, there exists an $\varepsilon$ such that the statement holds on $S \cap B_\tau(\varepsilon)$. 
Carathéodory condition

Definition

Let $\varphi \in S_2$. A point $\tau \in \mathbb{T}^2$ is a carapoint for $\varphi$ if there exists a sequence $\{\lambda_n\} \subset \mathbb{D}^2$ tending to $\tau$ such that

$$\frac{1 - |\varphi(\lambda_n)|}{1 - \|\lambda_n\|}$$

is bounded.

We also say that $\varphi$ satisfies the Carathéodory condition at $\tau$. 
Theorem (Agler, Mccarthy, Young)

The following are equivalent:

1. There exists a sequence \( \lambda_n \) tending to \( \tau \) in \( \mathbb{D}^2 \) such that
   \[
   \frac{1 - |\varphi(\lambda_n)|}{1 - \|\lambda_n\|} \text{ is bounded.}
   \]

2. There exists \( \omega \in \mathbb{T} \) such that for all nontangential \( S \), there exists \( \alpha > 0, \varepsilon > 0 \) such that for every \( \lambda \in S \cap B_\tau(\varepsilon) \),
   \[
   |\varphi(z) - \omega| \leq \alpha \|\tau - \lambda\|
   \]

If these conditions hold, then \( \varphi \) is directionally differentiable in every direction pointing into the bidisk at \( \tau \).
Theorem (Agler, McCarty, Young)

The following are equivalent:

1. There exist $\omega \in \mathbb{T}$ and $\eta \in \mathbb{C}^2$ such that for every nontangential set $S$ and every $\beta > 0$, there exists $\varepsilon > 0$ such that for all $\lambda \in S \cap B_\tau(\varepsilon)$,

$$|\varphi(\lambda) - \omega - \eta \cdot \lambda| \leq \beta \|\tau - \lambda\|$$

2. There exist $\omega \in \mathbb{T}$ and $\eta \in \mathbb{C}^2$ such that $\lim_{\lambda \xrightarrow{nt} \tau} \varphi(\lambda) = \omega$ and $\lim_{\lambda \xrightarrow{nt} \tau} \nabla \varphi(\lambda) = \eta$.

These conditions imply that the directional derivative $D_\delta \varphi(\tau)$ is linear in $\delta$. 
In short...

**B-point** \( \varphi \) has a \( B \)-point at \( \tau \) if and only if \( \varphi \) is directionally differentiable at \( \tau \) for all directions pointing into the bidisk at \( \tau \).

**C-point** \( \varphi \) has a \( C \)-point at \( \tau \) if \( \varphi \) has a nontangential linear approximation at \( \tau \).
**B- and C-points**

**Example**

\[ f(\lambda) = \frac{\lambda_1 + \lambda_2 - 2\lambda_1\lambda_2}{2 - \lambda_1 - \lambda_2}. \]

has a B-point that is not a C-point at (1, 1).

**Example**

\[ f(\lambda) = \frac{-4\lambda_1(\lambda^2)^2 + (\lambda^2)^2 + 3\lambda_1\lambda^2 - \lambda_1 + \lambda^2}{(\lambda^2)^2 - \lambda_1\lambda^2 - \lambda_1 - 3\lambda^2 + 4} \]

has a C-point at (1, 1).
Recent papers of, among others, Bickel, Knese, McCarthy, Pascoe, and Sola address the behavior of rational functions on the bidisk at boundary points.

These papers introduce different levels of regularity that fall between $B$– and $C$–points.
A $B^+$-point is a $B$-point at which the $\alpha$ in the condition

$$|\varphi(z) - \omega| \leq \alpha \|\tau - \lambda\|$$

is uniform for all choice of nontangential set $S$.

The authors exhibit a function that has a $B^+$ point that is not a $C$-point and also prove that if a rational function has a $B^+$-point at $\tau$, then $\tau$ is a $C$-point.
$B^J$-points


ϕ is inner if $ϕ : \mathbb{D}^2 \rightarrow \mathbb{C}$ has $|ϕ(τ)| = 1$ for almost every $τ \in \mathbb{T}^2$.

The authors develop a hierarchy of regularity of rational inner functions.

Loosely, a function $ϕ$ has a $B^J$-point at $τ$ if $ϕ$ has a carapoint at $τ$ and a conformally analogous function $h$ has a polynomial approximation to an order depending on $J$. 
Family of singular rational inner functions

Example

For a constant $y \in (0, 1)$, let $f_y$ be the function

$$f_y(\lambda) = \frac{\lambda^1 y + \lambda^2 (1 - y) - \lambda^1 \lambda^2}{1 - \lambda^1 (1 - y) - \lambda^2 y}.$$

For each $y$, $f_y$ has a singular $B$-point at $(1, 1)$ that is not a $C$-point.
Let $Y$ be a positive contraction on a Hilbert space $\mathcal{H}$. By the spectral theorem,

$$Y = \int y \, dE(y).$$

Let $I_Y(\lambda)$ be the $B(\mathcal{H})$-operator valued function

$$I_Y(\lambda) = \int f_y(\lambda) \, dE(y) = \frac{\lambda^1 Y + \lambda^2 (1 - Y) - \lambda^1 \lambda^2}{1 - \lambda^1 (1 - Y) - \lambda^2 Y}.$$
The function $I_Y : \mathbb{D}^2 \to B(H)$ is a rational inner function, in the sense that $\|I_Y(\lambda)\| < 1$ for $\lambda \in \mathbb{D}^2$ and $\|I_Y(\tau)\| = 1$ for all $\tau \in \mathbb{T}^2$.

$I_Y$ records the singular structure of the family $f_y$. 
## Definition

Let $\varphi$ be in $S_2$. The triple $(\mathcal{M}, u, I_Y)$ is a generalized model of $\varphi$ at $\chi = (1, 1)$ if

- $\mathcal{M}$ is a separable Hilbert space,
- $u : D_2 \to \mathcal{M}$ is analytic such that the equation
  
  $$\varphi(\mu) \varphi(\lambda) = \langle (1 - I(\mu))^* I(\lambda), u(\lambda), u(\mu) \rangle$$

  holds for all $\lambda, \mu \in D_2$ where

  $$I_Y(\lambda) = \lambda Y + \lambda_2 (1 - Y) - \lambda_1 \lambda_2 (1 - Y) - \lambda_2 Y,$$

  where $Y$ is a positive contraction on $\mathcal{M}$.
A generalized Hilbert space model

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1 - \varphi(\mu) = \langle (1 - I(\mu))^* I(\lambda), u(\lambda), u(\mu) \rangle
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for all \( \lambda, \mu \in D_2 \) where \( I_Y(\lambda) \) is the operator valued map

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such that the equation

$$1 - \overline{\varphi(\mu)} \varphi(\lambda) = \left\langle \left(1 - I(\mu)^* I(\lambda) \right) u(\lambda), u(\mu) \right\rangle$$

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$$I_Y(\lambda) = \frac{\lambda^1 Y + \lambda^2 (1 - Y) - \lambda^1 \lambda^2}{1 - \lambda^1 (1 - Y) - \lambda^2 Y},$$

where $Y$ is a positive contraction on $\mathcal{M}$. 
Generalized models are continuous at carapoints

Theorem (T.D., ’16 and Agler, T.D., Young, ’12)

Let $\varphi \in S_2$. $\chi = (1, 1)$ is a carapoint for $\varphi \in S_2$ if and only if there exists a generalized model $(\mathcal{M}, u, I)$ of $\varphi$ such that $u$ extends continuously to $(1, 1)$ on nontangential sets.
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That is, we can always find a model function $u(\lambda)$ in a generalized model that extends continuously to the boundary. In other words, it makes sense to write

$$\lim_{\lambda \nt \to \chi} u(\lambda) = u(\chi).$$

We can use $u$ to probe the behavior of $\varphi$ at a singular carapoint.
A Pick function is an analytic function from the complex upper halfplane into itself.

**Theorem (Agler, McCarthy, Young and Agler, T.D., Young)**

Suppose that $\varphi \in S_2$ has a carapoint at $\chi$. Let $-\delta$ be any direction pointing into the bidisk at $\chi$. Then there exists a function $h$ so that $h$ and $-zh$ are in the Pick class so that

$$D_{-\delta} \varphi(\chi) = -\varphi(\chi) \delta_2 h\left(\frac{\delta_2}{\delta_1}\right).$$

Conversely, for a function $h$ with $h$ and $-zh$ in the Pick class, there exists $\varphi \in S_2$ with slope function $h$. 
Theorem (T.D., 16)

Let $\varphi \in S_2$ have a carapoint at $\chi$ and let $(\mathcal{M}, u, I_Y)$ be a continuous generalized model for $\varphi$ at $(1, 1)$. Then $u(1, 1) \perp \ker Y(1 - Y)$ if and only if $\varphi$ has a C-point at $\varphi$. 
The structure of the rational inner function $I_Y$ captures the differential structure of general Schur functions at singular carapoints. That is to say, the family

$$f_Y(\lambda) = \frac{\lambda^1 y + \lambda^2 (1 - y) - \lambda^1 \lambda^2}{1 - \lambda^1 (1 - y) - \lambda^2 y}$$

models regularity at carapoints.

$I_Y$ itself is intrinsic to $\varphi$: it arises by desingularizing a standard Hilbert space model for $\varphi$. 
Q: Can the rich regularity structure of rational inner functions be used to describe associated regularity of general Schur functions?

\[ 1 - \overline{\varphi(\mu)} \varphi(\lambda) = \left\langle \left( 1 - I_Y(\mu)^* I_Y(\lambda) \right) u(\lambda), u(\mu) \right\rangle \]
Thank you.